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Boson Representations of Symplectic Algebras

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A reduction of the boson representation of the algebra of the noncompact group $Sp(4k, R)$, $k>0$, to its subgroup $SU(k)$ is realized. The reduction scheme has two main branches: one through the totally symmetric unitary representations of the maximal compact subalgebra $u(2k)$; the other through the ladder representations of the noncompact subalgebra $u(k, k)$. Both reductions are accomplished by means of the same set of Hermitian operators, but taken in different order. The case of $k=3$, for the group $Sp(12, R)$, used in the interacting vector boson model, is discussed in more detail.

1. INTRODUCTION

Symplectic models recently have been extensively applied in the theory of nuclear structure (Raychev, 1972; Acherova et al., 1975; Rosensteel and Rowe, 1977a, b, 1980). In the work of Vanagas et al. (1975) and Georgieva et al. (1982) the symplectic group $Sp(12, R)$, introduced as a dynamical symmetry group of the collective motions in nuclei, emerges as a natural noncompact generalization of the group $U(6)$. Georgieva et al. (1982) suggested and developed an interacting vector boson model (IVBM). In this paper the boson representation of the $Sp(12, R)$ algebra is discussed, with the emphasis on the maximal compact subgroup $U(6)$ and the reduction chain (see also Georgieva et al. 1983):

$$\begin{array}{c} U(6) \supset SU(2) \times SU(3) \\ \qquad\qquad\qquad \cup \\ \qquad\qquad\qquad O(3) \end{array}$$

As suggested by Vanagas (1971), another possible application of the boson representation of $Sp(12, R)$ is the nonrelativistic three-body problem. It is well known that by introducing Jacobi coordinates the three-body

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problem can be reduced to the translationally invariant problem of two quasiparticles with coordinates $\mathbf{x}_\alpha$ and associated momenta $\mathbf{q}_\alpha$ $(\alpha = 1, 2)$. Using these variables, one can always introduce creation and annihilation operators whose real bilinear forms generate an algebra isomorphic to the $Sp(12, R)$ algebra.

The group $Sp(12, R)$ is of the type $Sp(4k, R)$, where $k > 0$ is an integer. In this case, on the one hand the group $U(2k)$ appears as a maximal compact subgroup, which contains the direct product $SU(2) \times SU(k)$ of the two mutually complementary subgroups. On the other hand, the so-called ladder representation of the noncompact group $U(k, k)$ acts in the space of the boson representation of the $Sp(4k, R)$ algebra. There exists a connection between this ladder representation and the boson representation of $U(2k)$, which is realized through the third generator T_3 of the multiplier $SU(2)$ of the already mentioned direct product. This operator is also the first Casimir operator of the group $U(k, k)$. Different aspects of this relationship have been studied in the cases $k = 1$ (Alhassid *et al.*, 1983) and $k = 2$ (Kibler and Negadi, 1983a, b, 1984).

In the present paper we show that both reduction chains

$$Sp(4k, R) \to U(2k) \to SU(2) \times SU(k) \to SU(k)$$

$$Sp(4k, R) \to U(k, k) \to U(k) \times U(k) \to SU(k)$$

are equally convenient for the description of the arising representations of the group $SU(k)$.

In the case of the group $Sp(12, R)$, in which we are particularly interested, this allows us to include the $U(3, 3)$ algebra, which in our opinion will enrich the physical content of the IVBM. The latter will be discussed in a following paper.

2. REDUCTION OF THE BOSON REPRESENTATION OF $Sp(4k, R)$

The boson representation of the $Sp(4k, R)$ algebra is introduced in a standard way, using creation $a^+_{\alpha i}$ and annihilation $a_{\alpha i}$ $(\alpha = 1, 2;\ i = 1, 2, \ldots, k)$ operators, which satisfy Bose commutation relations:

$$[a_{\alpha i}, a^+_{\beta j}] = \delta_{\alpha\beta}\delta_{ij}$$

(all other commutators are zero): They act in a Hilbert space $\mathscr{H}$ with a vacuum $|0\rangle$, so that

$$a_{\alpha i}|0\rangle = 0$$

The scalar product is chosen so that the operator $a^+_{\alpha i}$ is the Hermitian conjugate of $a_{\alpha i}$ (Anderson et al., 1967).

An orthonormal basis in $\mathscr{H}$ can be introduced in the following way:

$$|\boldsymbol{\pi}, \boldsymbol{\nu}\rangle = \prod_{i=1}^{k} \frac{(a_{1i}^{+})^{\pi_i}}{(\pi_i!)^{1/2}} \prod_{j=1}^{k} \frac{(a_{2j}^{+})^{\nu_j}}{(\nu_j!)^{1/2}} |0\rangle \tag{1}$$

where $\boldsymbol{\pi} \equiv (\pi_1, \pi_2, \ldots, \pi_k)$ and $\boldsymbol{\nu} \equiv (\nu_1, \nu_2, \ldots, \nu_k)$ run through all possible sets of k nonnegative integers. The operators

$$a_{\alpha i}^{+} a_{\beta j}^{+}, \, a_{\alpha i} a_{\beta j}, \, a_{\alpha i}^{+} a_{\beta j}, \qquad \alpha, \beta = 1, 2; \quad i, j = 1, \ldots, k \tag{2}$$

commute as the generators of $Sp(4k, R)$ and therefore they generate a representation (Barut and Raszka, 1977) of the algebra of this group—the so-called boson representation. It is reducible and decomposes into two irreducible representations. One of them acts in the space $\mathscr{H}_{+}$, spanned over the vectors (1) for which $n = \pi + \nu$ ($\pi = \sum_{i=1}^{k} \pi_i$ and $\nu = \sum_{i=1}^{k} \nu_i$) is even, and the other acts in $\mathscr{H}_{-}$, defined by the condition n odd, and

$$\mathscr{H} = \mathscr{H}_{+} \oplus \mathscr{H}_{-}$$

A reducible representation of the group $U(2k)$ acts in $\mathscr{H}$. Its Wayl generators are part of the operators (2), namely

$$a_{\alpha i}^{+} a_{\beta j}, \qquad \alpha, \beta = 1, 2; \quad i, j = 1, 2, \ldots, k$$

The first-order Casimir operator of the group $U(2k)$ is

$$\mathscr{N} = a_{\alpha i}^{+} a_{\alpha i} \tag{3}$$

(summation over repeated Greek or Latin indices is understood throughout). Obviously $\mathscr{N}$ acts on the basis vectors according to the rule

$$\mathscr{N} |\boldsymbol{\pi}, \boldsymbol{\nu}\rangle = n |\boldsymbol{\pi}, \boldsymbol{\nu}\rangle \tag{4}$$

It follows, then, that the subspaces $\mathscr{H}_{+}$ and $\mathscr{H}_{-}$ decompose into a direct sum of eigensubspaces of $\mathscr{N}$, defined by the condition $n =$ fixed:

$$\mathscr{H}_{+} = \bigoplus_{n} \mathscr{H}_{n}^{+}, \qquad \mathscr{H}_{-} = \bigoplus_{n} \mathscr{H}_{n}^{-}$$

In each of the spaces $\mathscr{H}_n$, a totally symmetric irreducible unitary representation (IUR) of $U(2k)$ is realized. These IURs will be denoted by $[n]_{2k}$. The representations of the two mutually complementary groups $SU(2)$ and $SU(k)$ $[SU(2) \times SU(k) \subset U(2k)]$ acting in $\mathscr{H}$ have the following Wayl generators:

$$a_{\alpha i}^{+} a_{\beta i} - \tfrac{1}{2} \delta_{\alpha\beta} \mathscr{N} \qquad \text{for } SU(2) \tag{5}$$

$$a_{\alpha i}^{+} a_{\alpha j} - (1/k) \delta_{ij} \mathscr{N} \qquad \text{for } SU(k) \tag{6}$$

For the group $SU(2)$ we introduce also the standard Hermitian generators:

$$T_m = \tfrac{1}{2} a_{\alpha i}^{+} (\sigma_m)_{\alpha\beta} a_{\beta i} \tag{7}$$

where σ_m, $m=1,2,3$, are the Pauli matrices. The representation of the $SU(2)$ algebra is expanded to a representation of the $U(2)$ algebra by adding the operator $\mathcal{N}$ to the generators (5) or (7). The addition of $\mathcal{N}$ to the generators (6) corresponds to a transition from $SU(k)$ to $U(k)$. It should be noted that the operator $\mathcal{N}$ is the first-order Casimir operator for the group $U(2k)$ as well as for the groups $U(2)$ and $U(k)$.

All the Casimir operators of $SU(k)$ can be expressed in terms of $\mathcal{N}$ and the second-order Casimir operator of $SU(2)$,

$$\mathbf{T}^2=\sum_{m=1}^{3} T_m^2 \tag{8}$$

which follows from the fact that the groups $SU(2)$ and $SU(k)$ are mutually complementary and the representations $[n]_{2k}$ are symmetric. This can be established directly by using the Okubo theorem, according to which in the case of $SU(n)$ the Casimir operators $C_k^{(n)}$, $k>n$, are functions of $C_2^{(n)},\ldots,C_n^{(n)}$ (see also Moshinsky, 1962). In particular, we have

$$C_2^{(k)}=2\mathbf{T}^2+(k-2)\mathcal{N}+\frac{(k-2)}{2k}\mathcal{N}^2 \tag{9}$$

Therefore, the IURs of the groups $SU(2)$, $SU(k)$, and $SU(2)\times SU(k)$ acting in a given space $\mathcal{H}_n$, $n=$ fixed, can be labeled by the eigenvalues $T(T+1)$ of the operator $\mathbf{T}^2$:

$$\begin{aligned} &T=n/2,\, n/2-1,\ldots,0 \quad &&\text{for } n \text{ even}\\ &T=n/2,\, n/2-1,\ldots,1/2 \quad &&\text{for } n \text{ odd}\end{aligned} \tag{10}$$

Thus when $n=$ fixed and $T=$ fixed, $2T+1$ equivalent representations of the group $SU(k)$ arise. Each of them is labeled by the eigenvalues of the operator T_3: $-T,-T+1,\ldots,T$. As a result we get a realization of the reduction scheme:

$$Sp(4k,R)\xrightarrow{\mathcal{N}} U(2k)\xrightarrow{\mathbf{T}^2} SU(2)\times SU(k)\xrightarrow{T_3} SU(k) \tag{11}$$

Hence, in the framework of the discussed boson representation of the $Sp(4k,R)$ algebra all possible irreducible representations of the group $SU(k)$ are determined uniquely through all possible sets of the eigenvalues of the Hermitian operators $\mathcal{N}$, $\mathbf{T}^2$, and T_3.

It turns out that the transition from $Sp(4k,R)$ to $SU(k)$ can be realized also through the algebra of the group $U(k,k)$. In this case the reduction is also carried out by use of the operators $\mathcal{N}$, $\mathbf{T}^2$, and T_3, but taken in a different order. This follows from the fact that a reducible unitary representation (Dothan et al., 1965; Todorov, 1966) of the $U(k,k)$ algebra with Wayl generators

$$a_{1i}^{+}a_{1j},\ a_{1i}^{+}a_{2j}^{+},\ -a_{2i}a_{1j},\ -a_{2i}a_{2j}^{+}, \qquad i,j=1,\ldots,k \tag{12}$$

acts in $\mathscr{H}$.

Obviously the boson representation of the $Sp(4k, R)$ algebra contains the generators (12). The first-order Casimir operator of $U(k, k)$ is

$$C_1^{(k,k)} = a_{1i}^{+}a_{1i} - a_{2i}a_{2i}^{+} \tag{13}$$

and does not differ essentially from the operator T_3:

$$T_3 = \tfrac{1}{2}C_1^{(k,k)} + k/2$$

T_3 acts on the basis vectors (1) in the following way:

$$T_3|\boldsymbol{\pi}, \boldsymbol{\nu}\rangle = \tfrac{1}{2}(\pi-\nu)|\boldsymbol{\pi}, \boldsymbol{\nu}\rangle \tag{14}$$

and consequently its eigensubspaces $\mathscr{H}_{\pi-\nu}$ are defined by the condition $\pi-\nu=\text{fixed}$.

It is easy to prove that each of the spaces $\mathscr{H}_{\pi-\nu}$ is invariant with respect to the ladder representation of $U(k, k)$ and does not contain any other invariant subspaces. In other words, an irreducible representation (a ladder) of the $U(k, k)$ algebra is induced in each $\mathscr{H}_{\pi-\nu}$. Moreover,

$$\mathscr{H} = \bigoplus_{\pi-\nu} \mathscr{H}_{\pi-\nu} \tag{15}$$

The next step in the reduction is realized by means of the operator $\mathcal{N}$, which in this case is related to a representation of the maximal compact subgroup $U(k)\times U(k)$ of the group $U(k, k)$. The Wayl generators of this representation are part of the operators (12):

$$a_{1i}^{+}a_{1j},\ a_{2i}a_{2j}^{+}, \qquad i,j=1,2,\ldots,k \tag{16}$$

From the action of the operator $\mathcal{N}$ on the basis vectors [see (4)] it follows that each space $\mathscr{H}_{\pi-\nu}$ decomposes into a direct sum:

$$\mathscr{H}_{\pi-\nu} = \bigoplus_{\pi+\nu} \mathscr{H}_{\pi\nu} \tag{17}$$

The spaces $\mathscr{H}_{\pi\nu}$ are the common eigensubspaces of T_3 and $\mathcal{N}$ and are defined by the conditions $\pi-\nu=\text{fixed}$, $\pi+\nu=\text{fixed}$ or $\pi=\text{fixed}$, $\nu=\text{fixed}$.

Following the same considerations as above, one can see that an IUR of $U(k)\times U(k)$ (a step) acts in each $\mathscr{H}_{\pi\nu}$.

Finally, the transition $U(k)\times U(k)\to SU(k)$ is obtained with the help of the operator $\mathbf{T}^2$. As a result, one gets again a full description of all possible IURs of $SU(k)$ acting in $\mathscr{H}$. Now the corresponding reduction scheme is of the type

$$Sp(4k, R) \xrightarrow{T_3} U(k, k) \xrightarrow{\mathcal{N}} U(k)\times U(k) \xrightarrow{\mathbf{T}^2} SU(k) \tag{18}$$

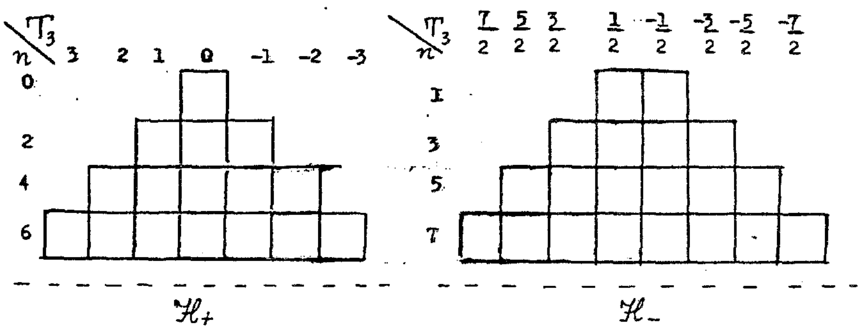

Fig. 1. The reductions $Sp(4k, R) \to U(k, k) \to U(k) \times U(k)$.
$\searrow U(2k)$

The splitting of the spaces $\mathscr{H}_+$ and $\mathscr{H}_-$ corresponding to the reductions $Sp(4k, R) \to U(k, k) \to U(k) \times U(k)$ and $Sp(4k, R) \to U(2k)$ is shown schematically in Figure 1, where the columns represent the separate ladders [the IURs of the $U(k, k)$ algebra, which are defined by the operator T_3] and the rows the UIRs of $U(2k)$ (defined by $\mathcal{N}$). Each cell corresponds to an IUR of $U(k) \times U(k)$.

The reduction $U(2k) \to SU(2) \times SU(k) \to SU(k)$ at $n =$ fixed is shown in Figure 2. Each of the rows represents an IUR of $SU(2) \times SU(k)$ assigned by the operator $\mathbf{T}^2$. The columns correspond to the IURs of $U(k) \times U(k)$. The cells represent the IUR of $SU(k)$, and each row contains $2T+1$ equivalent representations of this group, labeled by the eigenvalues of the operator T_3.

3. EXAMPLES

3.1. $k = 1$

This case is treated in Alhassid et al. (1983). In particular, it is shown there that if the boson realization is performed in terms of differential operators, then $U(2)$ and $U(1, 1)$ are groups generating the spectrum of the one-dimensional Schrödinger equation with Morse potential for bound and scattering states, respectively.

When $k = 1$ the operators $\mathcal{N}$, $\mathbf{T}^2$, and $T_3(C_1^{(1,1)})$ form a complete set of commuting Hermitian operators in $\mathscr{H}$.

3.2. $k = 2$

The importance of this case is underlined by the fact that one of the ladders of $U(2, 2)$ defined by $C_1^{(2,2)} = -2$ generates the whole spectrum of

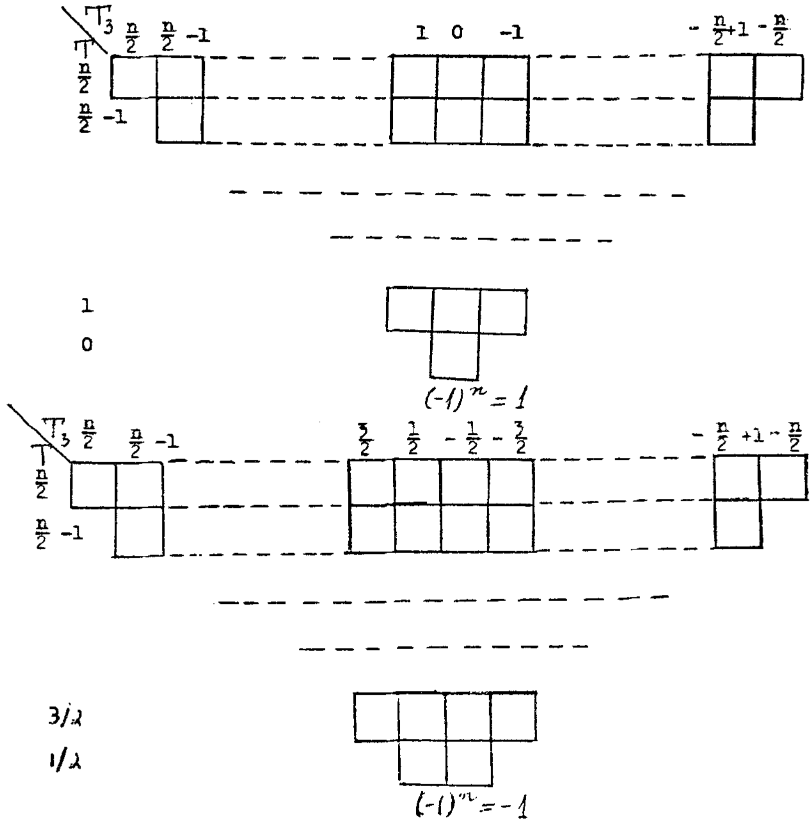

Fig. 2. The reduction $U(2k) \to SU(2) \times SU(k) \to SU(k)$.

the nonrelativistic hydrogen atom (Malkin and Man'ko, 1965; Barut and Kleinert, 1967; Mack and Todorov, 1969; Kibler and Negadi, 1983a, b; 1984). Kibler and Negadi (1983a, b, 1984) discuss the relationship between the nonrelativistic hydrogen atom and the four-dimensional harmonic oscillator with a symmetry group $U(4)$ in the framework of the boson representation of $Sp(8, R)$.

When $k = 2$, two mutually complementary $SU(2)$ groups appear. The first, with generators (7), will be denoted by $SU(2)_T$, and the second by $SU(2)_{T'}$. The Hermitian generators of the latter are

$$T'_m = \tfrac{1}{2} a^+_{\alpha i} (\sigma_m)_{ij} a_{\alpha j}, \qquad m = 1, 2, 3 \tag{19}$$

and [see (9)]

$$\mathbf{T}^2 = \tfrac{1}{2} C_2^{(2)} = \mathbf{T}'^2$$

The operators $\mathcal{N}$, $\mathbf{T}^2$, T_3 and T_3' form a complete set of commuting operators in the space $\mathcal{H}$.

3.3. $k=3$

This case will be considered in more detail. As already mentioned, Georgieva et al. (1982) developed IVBM based on the boson representation of $Sp(12, R)$. In that paper the variables are expressed in terms of cyclic coordinates. We shall rewrite some of the quantities discussed here in terms of the variables used in Georgieva et al. (1982). Within the IVBM, it is assumed that the collective motions in nuclei can be described by means of two types of elementary excitations—two types of vector bosons—which form a "pseudospin"-$SU(2)$ doublet.

Thus creation and annihilation operators are introduced:

$$\begin{aligned} u_m^+(\alpha) &= \frac{1}{\sqrt{2}}[x_m(\alpha) - iq_m(\alpha)] \\ u^m(\alpha) &= \frac{1}{\sqrt{2}}[x^m(\alpha) + iq^m(\alpha)] \end{aligned} \qquad \alpha = \pm 1/2, \quad m = 0, \pm 1 \tag{20}$$

where $x_m(\alpha)$ and $q_m(\alpha) = -i\,\partial/\partial x^m(\alpha)$ are the corresponding cyclic coordinates and momenta. [The metric tensor is $g_{mn} = g^{mn} = (-1)^n \delta_{m,-n}$.] The operators (20) satisfy the boson commutation relations

$$[u^m(\alpha), u_n^+(\beta)] = \delta_{\alpha,\beta}\delta_n^m$$

and Hermitian conjugation rules

$$[u_m^+(\alpha)]^+ = u^m(\alpha), \qquad [u^m(\alpha)]^+ = u_m^+(\alpha)$$

They act in a Fock space with a vacuum state

$$|0\rangle = \exp\left(-\frac{1}{2}\sum_{m=0,\pm 1} x^m x_m\right)$$

The correspondence between the operators $u_m^+(\alpha)$ and $u^m(\alpha)$ ($m = 0, \pm 1$; $\alpha = \pm 1/2$) and the operators $a_{\alpha i}^+$ and $a_{\alpha i}$ ($i = 1, 2, 3$; $\alpha = 1, 2$) is obvious.

The generators of the boson representation of the $Sp(12, R)$ algebra can now be written as

$$\begin{gathered} u_m^+(\alpha)u_n^+(\beta); \quad u^n(\alpha)u^m(\beta); \quad u_m^+(\alpha)u^n(\beta) \\ m, n = 0, \pm 1, \qquad \alpha, \beta = \pm 1/2 \end{gathered} \tag{21}$$

We also introduce the following notations:

$$p_m^+ = u_m^+(\alpha = \tfrac{1}{2}), \qquad n_m^+ = u_m^+(\alpha = -\tfrac{1}{2})$$

In terms of $\mathbf{p}^+$ and $\mathbf{n}^+$ the Wayl generators of the ladder representation of $U(3,3)$ are

$$p_k^+ p^l, \quad -n^l p^k, \quad p_k^+ n_l^+, \quad -n^l n_k^+, \qquad k, l = 0, \pm 1 \tag{22}$$

The Wayl generators of $U(6)$ and of the two mutually complementary groups $SU(3)$ and the "pseudospin" $SU(2)$, as well as of $U(3)\times U(3)$, can be expressed in an analogous way.

It is convenient to use a new set of generators of the $Sp(12, R)$ algebra. This set consists of irreducible tensor operators with respect to the $O(3)$ group:

$$\begin{aligned} F_M^L(\alpha,\beta) &= \sum_{k,m} C_{1k1m}^{LM} u_k^+(\alpha) u_m^+(\beta) \\ G_M^L(\alpha,\beta) &= \sum_{k,m} C_{1k1m}^{LM} u_k(\alpha) u_m(\beta) \\ A_M^L(\alpha,\beta) &= \sum_{k,m} C_{1k1m}^{LM} u_k^+(\alpha) u_m(\beta) \end{aligned} \tag{23}$$

where C_{1k1m}^{LM} are the Clebsch-Gordan coefficients for the decomposition $O(3)\supset O(2)$ (Varshalovich et al., 1975).

The operators $A_M^L(\alpha,\beta)$ $(\alpha,\beta=\pm\frac{1}{2}, L=0,1,2)$ generate an algebra of the maximal compact subgroup of $Sp(12, R)$, namely the group $U(6)$. Now the particle number operator $\mathcal{N}$ is

$$\mathcal{N} = -\sqrt{3}\sum_{\alpha} A^0(\alpha,\alpha) = \sum_{k=0,\pm 1} (p_k^+ p^k + n_k^+ n^k) \tag{24}$$

The $SU(3)$ subalgebra is generated by the operators

$$\tilde{Q}_M = \sqrt{6}\sum_{\alpha} A_M^2(\alpha,\alpha) \tag{25}$$

interpreted as a truncated (Eliott) quadrupole momentum, and by the components of the angular momentum

$$L_M = -\sqrt{2}\sum_{\alpha} A_M^1(\alpha,\alpha) \tag{26}$$

which define the $O(3)$ subalgebra.

The operators of the "pseudospin"

$$\begin{aligned} T_1 &= \sqrt{\tfrac{3}{2}} A^0(\tfrac{1}{2},-\tfrac{1}{2}); \qquad T_{-1} = -\sqrt{\tfrac{3}{2}} A^0(-\tfrac{1}{2},\tfrac{1}{2}) \\ T_3 = T_0 &= -\sqrt{\tfrac{3}{2}}[A^0(\tfrac{1}{2},\tfrac{1}{2}) - A^0(-\tfrac{1}{2},-\tfrac{1}{2})] \\ &= \frac{1}{2}\sum_{k=0,\pm 1} (p_k^+ p^k - n_k^+ n^k) \end{aligned} \tag{27}$$

define the algebra of the group $SU(2)\equiv SU(2)_T$.

The second-order Casimir operators of the group $SU(3)$

$$C_2^{(3)}=\tfrac{1}{6}\tilde{Q}^2+\tfrac{1}{2}\mathbf{L}^2$$

and of the group $SU(2)$

$$\mathbf{T}^2=\sum_{k=0,\pm 1}(-1)^k T_k T_{-k}$$

are related by [see (9)]

$$C_2^{(3)}=2\mathbf{T}^2+\tfrac{1}{6}\mathcal{N}^2+\mathcal{N}$$

When $n=$ fixed and $T=$ fixed, a total of $2T+1$ equivalent irreducible representations $[\lambda, \mu]_3$ of $SU(3)$ appear, where $\lambda=2T$ and $\mu=\frac{1}{2}n-T$.

It should be noted that the final reduction

$$SU(3)\supset SO(3)\supset O(2)$$

is performed in the standard way. As shown in Georgieva et al. (1983), the basis of Bargmann and Moshinsky (1961) is an appropriate one for the eigenvalue problem in IVBM.

Thus, in the framework of the boson representation of $Sp(12, R)$ all possible representations of $SU(3)$ are uniquely defined by all possible eigenvalues of the Hermitian operators $\mathcal{N}$, $\mathbf{T}^2$, and T_3. Moreover, the transition from $Sp(12, R)$ to $SU(3)$ can be realized through the group $U(3,3)$. In this case the reduction is also carried out by means of the same set of operators, but taken in a different order.

In general the realized reduction of the boson representations of a $Sp(4k, R)$ algebra through the representations of the maximal compact subalgebra $U(2k)$, as well as through the ladder representations of $U(k, k)$, could lead to a unified description of both the discrete and the continuous spectra observed in many quantum mechanical systems.

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